Preimage algorithms for the Tillich-Zémor hash function

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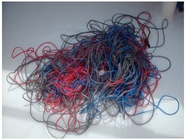
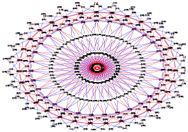
Hash functions and Cayley graphs

- Hash functions
  $H : \{0,1\}^* \rightarrow \{0,1\}^n$

- "Classical"
  hash functions



- Tillich-Zémor
  hash function



Tillich-Zémor hash function

- Mathematical structure : finite group, Cayley graph
- Proposed by Tillich-Zémor at CRYPTO'94 [TZ94] following previous (broken) scheme by Zémor [Z91]
- Trapdoor attack [SGGB00]
- Attacks on particular parameters [SGGB00,CP94,AK98]
- Until 13 months ago, best generic attacks were asymptotically inefficient [PQTZ08]

Tillich-Zémor hash function

- August'09 : very efficient collision attack
  by Grassl, Illic, Magliveras, Steinwandt [GIMS09]
- This paper : preimage algorithms
  (also very efficient)

Outline

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Tillich-Zémor hash function

Grassl et al.'s collision attack

Preimage algorithms

Conclusion

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Tillich-Zémor hash function

- $p \in \mathbb{F}_2[X]$ irreducible of degree n
  $K = \mathbb{F}_2[X]/(p(X)) \approx \mathbb{F}_{2^n}$
- Group $G = SL(2, K)$
  Generators $S = \{A_0 = \left(\begin{smallmatrix} X & 1 \\ 1 & 0 \end{smallmatrix}\right), A_1 = \left(\begin{smallmatrix} X & X+1 \\ 1 & 1 \end{smallmatrix}\right)\}$
- Message $m = m_1 ... m_N \in \{0, 1\}^N$

$$H(m_1 m_2 ... m_N) := A_{m_1} A_{m_2} ... A_{m_N} \bmod p(X)$$

Hard (?) problems

- **Balance problem :** ($\Leftrightarrow$ collisions)
  Given G and $S = \{s_0, ..., s_{k-1}\} \subset G$,
  find two short products $\prod s_{m_i} = \prod s_{m'_i}$
- **Representation problem :** ($\Rightarrow$ 2nd preimages)
  Given G and $S = \{s_0, ..., s_{k-1}\} \subset G$,
  find a short product $\prod s_{m_i} = 1$
- **Factorization problem :** ($\Leftrightarrow$ preimages)
  Given G, $g \in G$ and $S = \{s_0, ..., s_{k-1}\} \subset G$,
  find a short product $\prod s_{m_i} = g$

Outline

Changing the generators

- Let $A_0' := A_0^{-1}A_0A_0 = \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix}$,
  Let $A_1' := A_0^{-1}A_1A_0 = \begin{pmatrix} X+1 & 1 \\ 1 & 0 \end{pmatrix}$
- Let H' be H but replacing A_0, A_1 by A_0', A_1'
  $$H'(m) = A_0^{-1}H(m)A_0$$
- Collision for H' $\Leftrightarrow$ collision for H
- Preimage of g for H' $\Leftrightarrow$ preimage of $A_0gA_0^{-1}$ for H

! Notation : we write A_0, A_1, H instead of A_0', A_1', H'

Link with Euclidean algorithm

- $A_0 = \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} X+1 & 1 \\ 1 & 0 \end{pmatrix}$ are "Euclidean algorithm matrices"

$$a_{i-1} = q_i a_i + a_{i+1} \Leftrightarrow \begin{pmatrix} a_i & a_{i-1} \end{pmatrix} = \begin{pmatrix} a_{i-1} & a_{i-2} \end{pmatrix} \begin{pmatrix} q_i & 1 \\ 1 & \end{pmatrix}$$

- Let h be H "without modular reductions"

$$h(m_1...m_n) := A_{m_1}...A_{m_N}$$

- $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = h(m) \Rightarrow$ the Euclidean algorithm applied to (a, b) **only produces quotients** X **and** $X+1$

Mesirov and Sweet's algorithm

- **Theorem** [**MS87**] **:** for any **irreducible** $a \in \mathbb{F}_2[X]$, there exists $b \in \mathbb{F}_2[X]$ such that all quotients obtained by applying the Euclidean algorithm to (a, b) belong to $\{X, X+1\}$
- The proof is constructive

Building the collision

- Let p be the polynomial defining the field in TZ hash function
- Apply [MS87] to $a = p$: we obtain b and a message $m = m_1 \ldots m_N$ such that $H(m) = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$
- Swap the first bit

$$H(\bar{m}_1 m_2 \ldots m_N) = \begin{pmatrix} c & b+d \\ c & d \end{pmatrix}$$

- Build the palindrome $\tilde{m} = m_N \ldots m_2 \bar{m}_1 \bar{m}_1 m_2 \ldots m_N$

$$H(\tilde{m}) = \begin{pmatrix} 0 & 1 \\ 1 & b^2 \end{pmatrix}$$

- Observe collision

$$A_0 H(\tilde{m}) A_0 = A_1 H(\tilde{m}) A_1$$

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Second preimages

- Apply [MS87] to $a = p$: we obtain a message $m = m_1 \dots m_N$ such that $H(m) = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$
- Build the palindrome $\tilde{m} = m_N \dots m_2 \bar{m}_1 \bar{m}_1 m_2 \dots m_N$
- Observe
  - $H(0\tilde{m}) = \begin{pmatrix} 1 & X+b^2 \\ 0 & 1 \end{pmatrix}$ and $H(\tilde{m}0) = \begin{pmatrix} 1 & 0 \\ X+b^2 & 1 \end{pmatrix}$
  - Both matrices have order 2
    $\Rightarrow H(0\tilde{m}0\tilde{m}) = H(\tilde{m}0\tilde{m}0) = I$
- **Preimage of I** $\Rightarrow$ **second preimages** for any message
  $H(m_0) = I \quad \Rightarrow H(mm_0) = H(m_0 m) = H(m)$

Preimage algorithm

- **Precompute** preimages of $\begin{pmatrix} 0 & b_i \\ c_i & d_i \end{pmatrix}$ such that the set $\{b_i^2 + X\}$ is a basis of $\mathbb{F}_{2^n}/\mathbb{F}_2$
- Let $m = m_1 ... m_N$ such that $H(m) = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$. Then $H(\tilde{m}0) = \begin{pmatrix} 1 & 0 \\ X+b^2 & 1 \end{pmatrix}$ and $H(0\tilde{m}) = \begin{pmatrix} 1 & X+b^2 \\ 0 & 1 \end{pmatrix}$
- The "red matrices" belong to **Abelian subgroups** $\begin{pmatrix} 1 & 0 \\ \sum \alpha_i & 1 \end{pmatrix} = \prod \begin{pmatrix} 1 & 0 \\ \alpha_i & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \sum \beta_i \\ 0 & 1 \end{pmatrix} = \prod \begin{pmatrix} 1 & \beta_i \\ 0 & 1 \end{pmatrix}$ Write any α, β in the basis $\{b_i^2 + X\}$ using linear algebra
- Any matrix can be written as $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix}^\delta \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, $\delta \in \{0, 1\}$.

First precomputing algorithm

- **Goal :** obtain n messages hashing to matrices $\begin{pmatrix} 0 & b_i \\ c_i & d_i \end{pmatrix}$ such that the set $\{b_i^2 + X\}$ is a basis of $\mathbb{F}_{2^n}/\mathbb{F}_2$
  Applying [MS87] to $a = p$ we obtain one such message
- **Idea :** apply [MS87] to $a = pp'_i$ where p'_i small degree
- **Issue :** [MS87] requires a irreducible

First precomputing algorithm

- **We extend [MS87] :** Let p, p' be nonlinear irreducible polynomials and let $a = pp'$. If
$$\deg\left([X(X+1)p']^{-1} \bmod p\right) \leq \deg(p) - 2$$
then the Mesirov-Sweet's algorithm provides b such that all quotients computed by the Euclidean algorithm applied to (a, b) belong to $\{X, X+1\}$
- **Heuristic arguments + experiments :**
  - Small $\deg(p_i')$ suffice
  - Preimages of length $O(n^2)$ for TZ
  - Probabilistic time $O(n^4)$

Second precomputing algorithm

- **Goal :** obtain n messages hashing to matrices $\begin{pmatrix} 0 & b_i \\ c_i & d_i \end{pmatrix}$ such that the set $\{b_i^2 + X\}$ is a basis of $\mathbb{F}_{2^n}/\mathbb{F}_2$

  Applying [MS87] to $a = p$ we obtain one such message m_1

- **Idea :** build those messages recursively
  - Define $m_i := m_{i-1} 0 m_1$
  - **We prove** that $H(m_i) = \begin{pmatrix} 0 & b_1^i \\ c_i & d_i \end{pmatrix}$ for some c_i, d_i

- Do the elements $b_i^2 + X$ generate a basis of $\mathbb{F}_{2^n}/\mathbb{F}_2$?

Second precomputing algorithm

- **We prove :** If the minimal polynomial of b_1 has degree n, then we can extract a basis from $\{b_i^2 + X, i = 1, ..., 2n\}$
- When n is prime : always succeeds
  - Preimage of length $O(n^3)$ for TZ
  - Deterministic time $O(n^3)$
- When n is not prime
  - Succeeds with very high probability, same complexities (the analysis is partially heuristic)
  - Always succeeds in practice
  - (Other attacks exist)

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Preimage algorithms for TZ hash function

- Preimages in time $O(n^3)$ given some precomputation
- First precomputing algorithm :
  - Preimages of length $O(n^2)$ in probabilistic time $O(n^4)$
- Second precomputing algorithm :
  - Preimages of length $O(n^3)$ in deterministic time $O(n^3)$
  - Full proof when n is prime
- The case n prime proves a conjecture of Babai [BS92] for those particular parameters

Hash functions and Cayley graphs : the end of the story ?

- Similar functions have been broken as well (Zémor, LPS, Morgenstern)
- However, all these functions used very special parameters in a sense
- Strong connections with well-known problems in graph theory and group theory, with many applications in computer science (expander graphs...)
- Next challenge : $SL(2, \mathbb{F}_{2^n})$ with $A_0 = \begin{pmatrix} t_0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} t_1 & 1 \\ 1 & 0 \end{pmatrix}$ and $t_0 + t_1 \neq 1$

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